Chips Go BOOM BOOM!!!

Matvey Borodin, Hannah Han, Kaylee Ji, Alexander Peng, David Sun, Isabel Tu, Jason Yang, William Yang, Kevin Zhang, Kevin Zhao
Mentor: Tanya Khovanova

MIT PRIMES Step

5/20, 2018

Exploding Dots

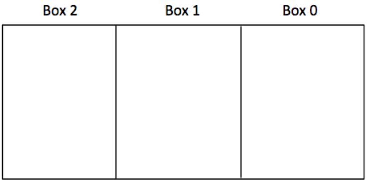
- Invented by James Tanton

Exploding Dots

- Invented by James Tanton
- Cool way of imagining how to represent numbers in fractional bases greater than 1

Exploding Dots

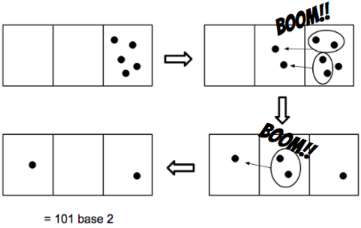
- Invented by James Tanton
- Cool way of imagining how to represent numbers in fractional bases greater than 1
- We have a row of boxes, which extend to the left, each representing a place digit holder, starting with box 0



- If we want to convert x into a fractional base $\frac{m}{n}$, we place x dots into box 0

- If we want to convert x into a fractional base $\frac{m}{n}$, we place x dots into box 0
- We explode m dots in box 0 and replace those m dots with n dots in box 1

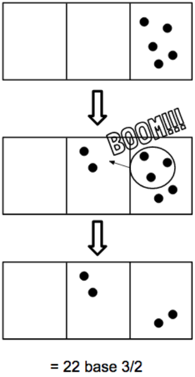
- If we want to convert x into a fractional base $\frac{m}{n}$, we place x dots into box 0
- We explode m dots in box 0 and replace those m dots with n dots in box 1
- We then do the same thing in box 1, and this continues until no more "explosions" can occur



- The number of dots remaining in each box represent the digit of the base $\frac{m}{n}$ number

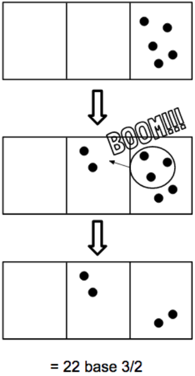
Base $\frac{3}{2}$

- We use the concept of exploding dots to represent numbers in base $\frac{3}{2}$



Base $\frac{3}{2}$

- We use the concept of exploding dots to represent numbers in base $\frac{3}{2}$



- Natural numbers in base $3/2$: $0, 1, 2, 20, 21, 22, 210, 211, 212, 2100, 2101, 2120...$ [OEIS Sequence A024629]

Properties of Base $\frac{3}{2}$

- $0, 1, 2, 20, 21, 22, 210, 211, 212, 2100, 2101, 2120\ldots$
- Every integer only uses the digits 0, 1, 2

Properties of Base $\frac{3}{2}$

- $0, 1, 2, 20, 21, 22, 210, 211, 212, 2100, 2101, 2120...$
- Every integer only uses the digits 0, 1, 2
- Every number starting with 2 in base 10 starts with 2 in base $\frac{3}{2}$

Properties of Base $\frac{3}{2}$

- $0, 1, 2, 20, 21, 22, 210, 211, 212, 2100, 2101, 2120 \ldots$
- Every integer only uses the digits 0, 1, 2
- Every number starting with 2 in base 10 starts with 2 in base $\frac{3}{2}$
- Every number starting with 8 in base 10 starts with 21 and is preceded by a 2 or 0

Properties of Base $\frac{3}{2}$

- $0, 1, 2, 20, 21, 22, 210, 211, 212, 2100, 2101, 2120...$
- Every integer only uses the digits 0, 1, 2
- Every number starting with 2 in base 10 starts with 2 in base $\frac{3}{2}$
- Every number starting with 8 in base 10 starts with 21 and is preceded by a 2 or 0
- The last digit repeats in a cycle of 3, the last two digits repeat in a cycle of 9, and so on: the last k digits repeat in a cycle of 3^k

Properties of Base $\frac{3}{2}$

- $0, 1, 2, 20, 21, 22, 210, 211, 212, 2100, 2101, 2120...$
- Every integer only uses the digits 0, 1, 2
- Every number starting with 2 in base 10 starts with 2 in base $\frac{3}{2}$
- Every number starting with 8 in base 10 starts with 21 and is preceded by a 2 or 0
- The last digit repeats in a cycle of 3, the last two digits repeat in a cycle of 9, and so on: the last k digits repeat in a cycle of 3^k
- Removing one or several last digits of an integer in this base gives another integer in the base.

Properties of Base $\frac{3}{2}$

- $0, 1, 2, 20, 21, 22, 210, 211, 212, 2100, 2101, 2120...$
- Every integer only uses the digits 0, 1, 2
- Every number starting with 2 in base 10 starts with 2 in base $\frac{3}{2}$
- Every number starting with 8 in base 10 starts with 21 and is preceded by a 2 or 0
- The last digit repeats in a cycle of 3, the last two digits repeat in a cycle of 9, and so on: the last k digits repeat in a cycle of 3^k
- Removing one or several last digits of an integer in this base gives another integer in the base.
- **Note how numbers in base 6/4 aren't the same as numbers in base 3/2, since those can contain a 3, 4, or 5 digit, which is why we call it base 3/2, not base 1.5**

Base $\frac{3}{2}$ vs Base $\frac{6}{4}$

0, 1, 2, 20, 21, 22, 210, 211, 212, 2100, 2101, 2120...
0, 1, 2, 3, 4, 5, 40, 41, 42, 43, 44, 45...

Base 1.5

- Numbers are represented using the digits 1, 0, H where H=0.5
- Natural numbers: 1, 1H, 1H0, 1H1, 1H0H, 1H10, 1H11

Isomorphism

Theorem

Every number in base 1.5 is the same as the number with 2 times its value in base 3/2 except with the digits 0, H, 1 replaced by 0, 1, 2 correspondingly.

Base 1.5

- Here are the numbers in ascending order 0, H, H0, 1, H00, HH, 10, H0H, H000, H1, HH0, 1H, H01, H00H, 100...

Base 1.5

- Here are the numbers in ascending order 0, H, H0, 1, H00, HH, 10, H0H, H000, H1, HH0, 1H, H01, H00H, 100...
- To create the sequence with numbers less than j digits we check all the numbers less than $0.5 * 1.5^{j-1}$

Base 1.5

- Here are the numbers in ascending order 0, H, H0, 1, H00, HH, 10, H0H, H000, H1, HH0, 1H, H01, H00H, 100...
- To create the sequence with numbers less than j digits we check all the numbers less than $0.5 * 1.5^{j-1}$
- The indices of the integers in the ascending sequence is 0,3,11,25,46,77,117...

Base 1.5

- Here are the numbers in ascending order 0, H, H0, 1, H00, HH, 10, H0H, H000, H1, HH0, 1H, H01, H00H, 100...
- To create the sequence with numbers less than j digits we check all the numbers less than $0.5 * 1.5^{j-1}$
- The indices of the integers in the ascending sequence is 0,3,11,25,46,77,117...
- If we place the numbers in order of the number of digits they have and then ordering them by value, we have a natural base order

Base 1.5

- Here are the numbers in ascending order 0, H, H0, 1, H00, HH, 10, H0H, H000, H1, HH0, 1H, H01, H00H, 100...
- To create the sequence with numbers less than j digits we check all the numbers less than $0.5 * 1.5^{j-1}$
- The indices of the integers in the ascending sequence is 0,3,11,25,46,77,117...
- If we place the numbers in order of the number of digits they have and then ordering them by value, we have a natural base order
- Numbers in natural base order,0, H, 1, H0, HH, H1, 10, 1H, 11, H00, H0H, H01, HH0, HHH, HH1...

Base 1.5

- Here are the numbers in ascending order 0, H, H0, 1, H00, HH, 10, H0H, H000, H1, HH0, 1H, H01, H00H, 100...
- To create the sequence with numbers less than j digits we check all the numbers less than $0.5 * 1.5^{j-1}$
- The indices of the integers in the ascending sequence is 0,3,11,25,46,77,117...
- If we place the numbers in order of the number of digits they have and then ordering them by value, we have a natural base order
- Numbers in natural base order,0, H, 1, H0, HH, H1, 10, 1H, 11, H00, H0H, H01, HH0, HHH, HH1...
- The indexes of the integers in this sequences is 0, 2, 7, 21, 23, 64, 69, 71, 193, 207..., which is sequence A265316 in the OEIS

A265316

- Start with the greedy partition of the whole numbers of avoiding 3-term arithmetic progressions

A265316

- Start with the greedy partition of the whole numbers of avoiding 3-term arithmetic progressions
- Whole numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

A265316

- Start with the greedy partition of the whole numbers of avoiding 3-term arithmetic progressions
- Whole numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...
- Create a sequence from this. Start with 0, and keep taking the next smallest number that does not form a 3-term arithm.-prog.

A265316

- Start with the greedy partition of the whole numbers of avoiding 3-term arithmetic progressions
- Whole numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...
- Create a sequence from this. Start with 0, and keep taking the next smallest number that does not form a 3-term arithm.-prog.
- $0 \to 0, 1 \to 0, 1, 3 \to 0, 1, 3, 4 \to 0, 1, 3, 4, 9 \to \ldots$

A265316

- Start with the greedy partition of the whole numbers of avoiding 3-term arithmetic progressions
- Whole numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...
- Create a sequence from this. Start with 0, and keep taking the next smallest number that does not form a 3-term arithm.-prog.
- $0 \to 0, 1 \to 0, 1, 3 \to 0, 1, 3, 4 \to 0, 1, 3, 4, 9 \to \ldots$
- Now remove all whole numbers in this sequence from the original sequence of whole numbers, and remaining numbers are 2, 5, 6, 7, 8 ...

A265316

- Start with the greedy partition of the whole numbers of avoiding 3-term arithmetic progressions
- Whole numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...
- Create a sequence from this. Start with 0, and keep taking the next smallest number that does not form a 3-term arithm.-prog.
- $0 \to 0, 1 \to 0, 1, 3 \to 0, 1, 3, 4 \to 0, 1, 3, 4, 9 \to \ldots$
- Now remove all whole numbers in this sequence from the original sequence of whole numbers, and remaining numbers are 2, 5, 6, 7, 8 ...
- Start with the smallest number here (in this example 2) and repeat

A265316 (continued)

- Resulting partitions:
- 0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30...
- 2, 5, 6, 11, 14, 15, 18, 29, 32, 33, 38...
- 7, 8, 16, 17, 19, 20, 34, 35, 43, 44, 46...
- 21, 22, 24, 25, 48, 49, 51, 52, 57, 58,60...
- 23, 26, 50, 53, 59, 62, 63, 66, 72, 75, 104...
- 64, 65, 67, 68, 73, 74, 76, 77, 145, 146, 148...

A265316 (continued)

- Resulting partitions:
- 0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30...
- 2, 5, 6, 11, 14, 15, 18, 29, 32, 33, 38...
- 7, 8, 16, 17, 19, 20, 34, 35, 43, 44, 46...
- 21, 22, 24, 25, 48, 49, 51, 52, 57, 58,60...
- 23, 26, 50, 53, 59, 62, 63, 66, 72, 75, 104...
- 64, 65, 67, 68, 73, 74, 76, 77, 145, 146, 148...
- Take the first number from each partition to get A265316:
- 0, 2, 7, 21, 23, 64...

Chips Firing

- One-player game where one "fires chips" left and right to points relative to the origin.
- All chips begin on the origin, written as the one's digit place.
- If there exists at least $a + b$ chips at any point, a chips are fired out into the point to the left, and b into the point to the right

Basic Chip Firing Lemmas

Lemma

The order in which vertices fire does not matter.

- We then find that the final representation of the starting firing is unique

Lemma

The total number of chips does not change.

- Every integer n has a unique representation as the total number of chips is equal to n.

a-a firing

- In this system, firing is symmetrical relative to the origin.

Lemma

The resulting string for a-a firing has the $n \bmod a$ chips at the origin. It will also have $\lfloor \frac{n}{2a} \rfloor$ points each with a chips on either side

1-y Firing

- Consider the example of 1-2 firing, where we fire 1 chip to the left, and 2 chips to the right. The numbers 0, 1, and 2 are represented as themselves with a dot.

4: 11.2
5: 12.2
6: 21.12
7: 22.12
8: 111.1112
9: 112.1112
10: 121.11112
11: 122.11112
12: 221.1111112

1-y Firing

- Consider the example of 1-2 firing, where we fire 1 chip to the left, and 2 chips to the right. The numbers 0, 1, and 2 are represented as themselves with a dot.
- Something interesting happens after 3:

4: 11.2
5: 12.2
6: 21.12
7: 22.12
8: 111.1112
9: 112.1112
10: 121.11112
11: 122.11112
12: 221.1111112

1-y Firing

- Consider the example of 1-2 firing, where we fire 1 chip to the left, and 2 chips to the right. The numbers 0, 1, and 2 are represented as themselves with a dot.
- Something interesting happens after 3:

4: 11.2
5: 12.2
6: 21.12
7: 22.12
8: 111.1112
9: 112.1112
10: 121.11112
11: 122.11112
12: 221.1111112

- None of the next numbers contain a zero

1-y Firing

- Consider the example of 1-2 firing, where we fire 1 chip to the left, and 2 chips to the right. The numbers 0, 1, and 2 are represented as themselves with a dot.
- Something interesting happens after 3:

4: 11.2
5: 12.2
6: 21.12
7: 22.12
8: 111.1112
9: 112.1112
10: 121.11112
11: 122.11112
12: 221.1111112

- None of the next numbers contain a zero
- The numbers to the left of the origin point behave similarly to binary

1-y Firing Lemmas

- We need to understand the following Lemma to prove this

Lemma

In 1-y firing, the string $X(y+1).(y-1)_d y$ fires to $(X+1)1.(y-1)_{d+1}y$, where X is any string

$X+1$ in this case means incrementing the last digit of X.
This leads us to the final state of 1-y chip firing

Lemma

Consider the final state of n in 1-y firing, where $n > y+1$. If $n = y+2$, the final representation of it is 11.y. After that the left part before the radix follows sequence S(y).
The right part consist of several digits y-1 followed by one digit y.
If the number of digits y that are at the end of the left part of n is c, then the amount of $y-1$'s after the radix is increased by c when transitioning from n to n+1.

2-3

- Fire 2 chips left, 3 chips right
- To remove pathological examples, we start at 6 6: 21.3
  7: 22.3
  8: 23.3
  9: 24.3
  10: 42.13
  11: 43.13
  12: 44.13
  13: 213.43

2-3

- Fire 2 chips left, 3 chips right
- To remove pathological examples, we start at 6 6: 21.3
  7: 22.3
  8: 23.3
  9: 24.3
  10: 42.13
  11: 43.13
  12: 44.13
  13: 213.43
- The left side behaves similarly to the base $\frac{3}{2}$ from earlier

The Big Theorem of a-b firing

- Denote a_n as the number of chips in the node n places left of the origin (negative n means it is to the right)

The Big Theorem of a-b firing

- Denote a_n as the number of chips in the node n places left of the origin (negative n means it is to the right)
- Consider the summation series

$$\sum a_n * t^n$$

for all integers n and any number t

The Big Theorem of a-b firing

- Denote a_n as the number of chips in the node n places left of the origin (negative n means it is to the right)
- Consider the summation series

$$\sum a_n * t^n$$

  for all integers n and any number t
- When the kth node has to fire a chips left and b chips right, this will change the amount of chips in the nodes from a_{k+1}, a_k, a_{k-1} to $a_{k+1} + a, a_k - (a + b), a_{k-1} + b$

The Big Theorem of a-b firing

- Denote a_n as the number of chips in the node n places left of the origin (negative n means it is to the right)
- Consider the summation series

  $$\sum a_n * t^n$$

  for all integers n and any number t
- When the kth node has to fire a chips left and b chips right, this will change the amount of chips in the nodes from a_{k+1}, a_k, a_{k-1} to $a_{k+1} + a, a_k - (a+b), a_{k-1} + b$
- The value of the series will change by $at^{k+1} - (a+b)t^k + bt^{k-1} = t^{k-1}(at-b)(t-1)$ This value is 0 when $t =$0, 1, or $\frac{b}{a}$

The Big Theorem of a-b firing

- Denote a_n as the number of chips in the node n places left of the origin (negative n means it is to the right)
- Consider the summation series

$$\sum a_n * t^n$$

  for all integers n and any number t
- When the kth node has to fire a chips left and b chips right, this will change the amount of chips in the nodes from a_{k+1}, a_k, a_{k-1} to $a_{k+1} + a, a_k - (a+b), a_{k-1} + b$
- The value of the series will change by $at^{k+1} - (a+b)t^k + bt^{k-1} = t^{k-1}(at-b)(t-1)$ This value is 0 when $t =$0, 1, or $\frac{b}{a}$
- Therefore when we fire N chips, its final form interpreted in base $\frac{b}{a}$ is also N

The Big Theorem of a-b firing

- Denote a_n as the number of chips in the node n places left of the origin (negative n means it is to the right)
- Consider the summation series

$$\sum a_n * t^n$$

  for all integers n and any number t
- When the kth node has to fire a chips left and b chips right, this will change the amount of chips in the nodes from a_{k+1}, a_k, a_{k-1} to $a_{k+1} + a, a_k - (a + b), a_{k-1} + b$
- The value of the series will change by $at^{k+1} - (a + b)t^k + bt^{k-1} = t^{k-1}(at - b)(t - 1)$ This value is 0 when $t =$0, 1, or $\frac{b}{a}$
- Therefore when we fire N chips, its final form interpreted in base $\frac{b}{a}$ is also N
- Example: 13 $\rightarrow$ 213.43 in 2-3 firing, which is 13 interpreted as base $\frac{3}{2}$

Acknowledgements

Special thanks to:
Our Mentor Tanya Khovanova for guiding us in our research.
Cause she's the mentor.
MIT PRIMES Program for providing us a workspace, resources, and everything